Predicate Calculus

Quantifier Equivalence Rules

Quantifiers, disjunctions and conjunctions

- Earlier we said that existentially quantified propositions are disjunctions ranging over all the objects in the universe of discourse.   In other words
  - $\exists$xFx means (Fa v Fb v Fc v Fd v etc.)
- Similarly, universally quantified propositions are conjunctions over all the objects in the universe.
  - $\forall$xFx means (Fa & Fb & Fc & Fd & etc.)
- You also know that the De Morgan's equivalences establish that every disjunction is equivalent to a certain conjunction, and that every conjunction is equivalent to a certain disjunction.
- So it may not surprise you that we have rules that establish equivalences between existentially and universally quantified propositions as well. These are called the Quantifier Equivalence Rules.

Quantifier Equivalence rules (QE)

- There are 4 different quantifier equivalence rules, but they are all variations on the same theme, and they all have the same abbreviation: QE.

- We will introduce these rules by using specific examples involving the predicate F and the variable x. The rules work the same no matter what sort of predicate or relation is involved.

- Like De Morgan's all of the QE rules express one kind of formula as a negated version of the other.

- Also, just as with the other equivalences we are not restricted to working on the main operator. We can substitute equivalent expressions within formulas.

$\sim\exists xFx \leftrightarrow \forall x{\sim}Fx$

- The first QE rule says

  $$\vdash \sim\exists xFx \leftrightarrow \forall x{\sim}Fx$$

- Intuitively: "It is not the case that something is a fish" means "Everything is not a fish."

- The feeling of using a QE rule is identical to the feeling of using the DeMorgans rule. Moving the negation past the existential changes it to the universal, and vice versa.

$\sim\forall xFx \leftrightarrow \exists x\sim Fx$

- The second QE rule says

$$\vdash \sim\forall xFx \leftrightarrow \exists x\sim Fx$$

- Intuitively: "It is not the case that everything is a fish" means "Something is not a fish."

$\exists xFx \leftrightarrow \sim\forall x\sim Fx$

- The third QE rule is

  $\vdash \exists xFx \leftrightarrow \sim\forall x\sim Fx$

- Intuitively: "Something is a fish" means "It is not the case that nothing is a fish."

- Note that the 3rd rule can actually be derived using the 1st rule. Here is the derivation in one direction only.

- 1. | $\exists xFx$ H
- 2. | $\sim\sim\exists xFx$ 1, DN
- 3. | $\sim\forall x\sim Fx$ 2, QE
- 4. $\exists xFx \rightarrow \sim\forall x\sim Fx$ 1-3 $\rightarrow$I

- Notice how the negation that is the main operator in 2 is not involved. So here we are taking advantage of the ability to operate on a part of a formula.

$\forall xFx \leftrightarrow \sim\exists x\sim Fx$

- Finally, the fourth QE rules says:

  $\vdash \forall xFx \leftrightarrow \sim\exists x\sim Fx$

- Intuitively: "Everything is a fish" means "It is not the case that something is not a fish."

- The fourth QE rule is easily derived from the 2nd. Here is the derivation in one (the reverse) direction.

- 1. | $\sim\exists x\sim Fx$ H
- 2. | $\sim\sim\forall xFx$ 1, QE
- 3. | $\forall xFx$ 2, DN
- 4. $\sim\exists x\sim Fx \rightarrow \forall xFx$ 1-3 $\rightarrow$I

Summary

- Using the metalanguage the formal statement of the Quantifier Equivalence Rule is:

- If for some variable $\mathcal{x}$ and some partial formula $\mathcal{P}$ containing $\mathcal{x}$, $\forall \mathcal{x}\mathcal{P}$ or $\exists \mathcal{x}\mathcal{P}$ occur either as complete formulas or as wffs. within a larger formula $\mathcal{R}$, we may validly substitute one member of any of the following pairs of formulas for the other within $\mathcal{R}$:

  - $\sim\exists \mathcal{x}\mathcal{P}$ and $\forall \mathcal{x}\sim\mathcal{P}$
  - $\sim\forall \mathcal{x}\mathcal{P}$ and $\exists \mathcal{x}\sim\mathcal{P}$
  - $\exists \mathcal{x}\mathcal{P}$ and $\sim\forall \mathcal{x}\sim\mathcal{P}$
  - $\forall \mathcal{x}\mathcal{P}$ and $\sim\exists \mathcal{x}\sim\mathcal{P}$

Example 1

- Prior to having QE, we would have had to prove the following result indirectly.
- ~∃x∃yFxy ├ ~Fab
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- 1. ~∃x∃yFxy A
- 2. ∀x~∃yFxy 1, QE
- 3. ~∃yFay 2, ∀E
- 4. ∀y~Fay 3, QE
- 5. ~Fab 4, ∀E

Example 2

$\vdash \exists x \exists y Fxy \rightarrow \sim\forall x \forall y \sim Fxy$

1.	$\exists x \exists y Fxy$	H for $\rightarrow$I
2.	$\sim\forall x \sim\exists y Fxy$	1, QE
3.	$\sim\forall x \forall y \sim Fxy$	2, QE
4.	$\exists x \exists y Fxy \rightarrow \sim\forall x \forall y \sim Fxy$	1-3 $\rightarrow$I

Example 3

- ∀x~Fxc → ∃xGxb ⊢ ∃x(~Fxc →Gxb)

1.	∀x~Fxc → ∃xGxb	A
2.	~∃x(~Fxc → Gxb)	H for ~I
3.	∀x~(~Fxc → Gxb)	2, QE
4.	~(~Fac → Gab)	3, ∀E
5.	~Fac & ~Gab	4, ~→
6.	~Fac	5, &E
7.	∀x~Fxc	6, ∀I
8.	∃xGxb	1,7 →E
9.	Gab	H
10.	~Gab	5, &E
11.	P & ~P	9,10 CONT
12.	P & ~P	8, 9-11 ∃E
13.	~~∃x(~Fxc → Gxb)	2-12 ~I
14.	∃x(~Fxc → Gxb)	13, DN

Remember this is a restricted move. We have to make sure that the name we are introducing the variable for isn't in any assumption or live hypothesis.

Again it is ok to use 'a' here because even though it occurs above, it is not in a hypothesis or an assumption.

This trick is necessary, because we would not be able to discharge the hypothesis if 'a' were part of the contradiction.