Predicate Calculus

Rules for the Existential Quantifier

Existential Introduction.

- Existential introduction is the unrestricted rule. It permits you to introduce an existential quantifier for any name whatsoever. Here is a simple examples of its proper use.

Fab $\vdash \exists y \exists x Fyx$

1.	Fab	A
2.	$\exists x Fax$	1, $\exists$I
3.	$\exists y \exists x Fyx$	2, $\exists$I

Existential Introduction.

- The intuitive justification of existential introduction is straightforward.

- If you know, for example, that Albert is funny, then it follows that at least one thing is funny.

- If you know that Albert is funnier than Bart, then you know that there is at least one thing that is funnier than at least one thing.

Existential Introduction

- Because existential introduction is unrestricted, it is not necessary to introduce the variable for every occurrence of the name. For example, this is a perfectly good proof.

Laa ├ ∃y∃xLyx

1.	Laa	A
2.	∃xLax	1, ∃I
3.	∃y∃xLyx	2, ∃I

- To understand intuitively why step 2 is ok, you just need to remember that the variables don't inherently refer to different objects. If, for example, Alvin loves himself, then it does follow that there exists a y and an x such that y loves x. It just so happens that in this case Alvin is satisfying both variables.

Formal Statement of rule of ∃I

- The formal statement of the rule for ∃I is as follows.

- Rule of existential introduction

  - For a formula $\mathcal{P}$ containing a name a we may infer any formula of the form $\exists x\mathcal{P}$, where the variable x is inserted for a in $\mathcal{P}$.

Existential Elimination.

- The rule of existential elimination is another method of hypothetical proof.

- What this means is that if you have an existentially quantified expression like $\exists xFx$, you can hypothesize , say Fa, by indenting the line on which the hypothesis occurs and identifying it with an H.

- As with the other hypothetical rules, your proof will not be finished until you discharge this hypothesis from the proof.

Example of Existential Elimination.

- Here is a partial example of a properly done ∃E proof.
- $\forall x(Fx \rightarrow Gx), \exists x\, Fx \vdash \exists x Gx$

1.	$\forall x(Fx \rightarrow Gx)$	A
2.	$\exists x\, Fx$	A
3.	$\quad Fa$	H
4.	$\quad Fa \rightarrow Ga$	1, ∀E
5.	$\quad Ga$	3,4 →E
6.	$\quad \exists x Gx$	5, ∃I

- We have now reached our desired conclusion, but the hypothesis has not yet been discharged. To discharge the hypothesis you simply remove the indent, repeat the contents of line 6, and drop a line.

Example of Existential Elimination.

- $\forall x(Fx \rightarrow Gx), \exists x\, Fx \vdash \exists x Gx$

1.	$\forall x(Fx \rightarrow Gx)$	A
2.	$\exists x\, Fx$	A
3.	$\quad Fa$	H
4.	$\quad Fa \rightarrow Ga$	1, $\forall$E
5.	$\quad Ga$	3,4 $\rightarrow$E
6.	$\quad \exists x\, Gx$	5, $\exists$I
7.	$\exists x\, Gx$	2, 3-6, $\exists$E

- Like so. Notice that the justification contains the line on which the original existential formula occurred, and then all the lines involved in the derivation.

Restrictions on Existential Elimination 1

- The existential elimination rule is restricted for the same reason that the rule of universal introduction is restricted, viz., to insure that the name has been chosen arbitrarily.

- For example, if you know that someone is goofy, it obviously does not follow that Sam is goofy.

- The rule of ∃E is basically a trick that allows you to introduce a name and eliminate the quantifier so that you can use the inference rules and equivalences of propositional logic.

- The trick is to use a name that is essentially still a variable. We achieve this by making sure the name does not refer to any known individual.

Restrictions on Existential Elimination 2

- In the predicate calculus, what this restriction amounts to is that you must insure that the name you choose does not occur
  - 1. In any assumption; or
  - 2. In any live (i.e., undischarged) hypothesis.
- (These are exactly the same restriction we already observe for the rule of universal introduction.)

Restrictions on Existential Elimination 3

- However, there are two further restrictions on ∃E.
- 3. The name you use in your hypothesis must <u>not occur</u> in the original existential statement.
  - For example, you can not, for the purpose of an existential elimination proof involving ∃xFxa hypothesize Faa.
  - (Note: this sounds like we are going back on the general claim that when doing hypothetical proofs you can hypothesize anything you want. We aren't really. In the example above, you actually can hypothesize Faa; it's just that it won't be of any value to you in an existential elimination proof.)
- 4. The name you introduce in the existential elimination hypothesis must not <u>remain</u> in the final conclusion of the existential derivation.

Example of bad Existential Elimination.

- Here is an example of an improperly done ∃E proof.

∃xFx & ∃xGx, ⊢ ∃x(Fx & Gx)

1.	∃xFx & ∃xGx	A
2.	∃x Fx	1, &E
3.	∃x Gx	1, &E
4.	Fa	H (for ∃E)
5.	Ga	H (for ∃E)

- Line 5 is a bad hypothesis for ∃E, because 'a' already occurs in the live hypothesis above it, which violates the 2nd restriction.

Example of bad Existential Elimination 2.

- Here is another example of an improperly done ∃E proof.
- ∃x∃yFxy ├ ∃xFxa

1. ∃x∃yFxy	A
2. ∃yFby	H (for ∃E)
3. | Fba	H (for ∃E)
4. | ∃xFxa	3, ∃I
5. ∃xFxa	2, 3-4 ∃E

- Step 5 is not permissible because the name 'a' is what was hypothesized in line 3. (Note: The problem is not that 'a' occurs in the hypothesis, it is that 'a' was the very name that was introduced in the hypothesis.) This violates the 4^{th} restriction.

Example of bad Existential Elimination 3.

- Here is one final example of an improperly done ∃E proof.

$\forall x\ \exists y(Fx \lor Gyx) \vdash \exists y(Fy \lor Gyy)$

1.	$\forall x\ \exists y(Fx \lor Gyx)$	A
2	$\exists y(Fa \lor Gya)$	1, ∀E
3.	│ $Fa \lor Gaa$	H (for ∃E)
4.	│ $\exists y(Fy \lor Gyy)$	3, ∃I
5.	$\exists y(Fy \lor Gyy)$	2, 3-4 ∃E

- Step 5 is not permissible because 'a' was already in the formula on line 2, violating the 3rd restriction. This mistake 'feels' like it was made on line 3, but it doesn't technically occur until you try to discharge the hypothesis with an existential elimination proof..

Formal statement of the rule of ∃E

- Here is the formal statement of the rule of ∃E.

- <u>Rule of existential elimination</u>
  - Given an existentially quantified formula $\exists x \mathcal{P}$ and a derivation of some conclusion $\mathcal{R}$ from a hypothesis that results from removing $\exists x$ and inserting a single name letter a for every occurrence of x in $\mathcal{P}$, we may discharge the hypothesis and assert $\mathcal{R}$, provided that the name letter a does not occur in
    - i. any assumption
    - ii. any undischarged hypothesis
    - iii. $\exists x \mathcal{P}$
    - iv. $\mathcal{R}$

Another good existential elimination proof

$\exists x(Fx \rightarrow Gx), \forall xFx \vdash \exists xGx$

1.	$\exists x(Fx \rightarrow Gx)$	A
2.	$\forall xFx$	A
3.	| $Fa \rightarrow Ga$	H (for $\exists$E)
4.	| Fa	2, $\forall$E
5.	| Ga	3,4 $\rightarrow$E
6.	| $\exists xGx$	5, $\exists$I
7.	$\exists xGx$	1, 3-6 $\exists$E

An attempt at an existential elimination proof

$\forall x(Fx \rightarrow Gx), \exists x{\sim}Gx \vdash {\sim}\forall xFx$

1. $\forall x(Fx \rightarrow Gx)$ A
2. $\exists x{\sim}Gx$ A
3. $Fa \rightarrow Ga$ 1, $\forall$E
4. ${\sim}Ga$ H for $\exists$E
5. ${\sim}Fa$ 3,4 MT
6. ?

- Here we are stuck. We can not do a $\forall$I on 5, because 'a' is in a live hypothesis. And even if we could, we would only get $\forall x{\sim}Fx$, which is not what we are looking for. Whenever you are stuck like this you should consider doing a ~I. We could do that on line 6, but let's just start over.

A successful approach

$\forall x(Fx \rightarrow Gx), \exists x{\sim}Gx \vdash {\sim}\forall xFx$

1.	$\forall x(Fx \rightarrow Gx)$	A
2.	$\exists x{\sim}Gx$	A
3.	$\forall xFx$	H for ~I
4.	Fa	3, ∀E
5.	$Fa \rightarrow Ga$	1, ∀E
6.	Ga	4,5 →E
7.	${\sim}Ga$	H for ∃E
8.	$Ga \,\&\, {\sim}Ga$	6,7 &I
9.	?	

- We got our contradiction, but now what? We can not simply re-write "Ga & ~Ga" and discharge our hypothesis because 'a' is in the hypothesis, and that violates the 2^{nd} restriction.
- There is a neat trick for dealing with this situation. The contradiction rule allows you to write anything you want if you have a contradiction. What we want is a contradiction without 'a' in it. So on line 7 we can simply write "P & ~P"

A successful approach 2

$\forall x(Fx \rightarrow Gx), \exists x{\sim}Gx \vdash {\sim}\forall xFx$

1.	$\forall x(Fx \rightarrow Gx)$	A
2.	$\exists x{\sim}Gx$	A
3.	$\vert$ $\forall xFx$	H for ~I
4.	$\vert$ Fa	3, $\forall$E
5.	$\vert$ $Fa \rightarrow Ga$	1, $\forall$E
6.	$\vert$ Ga	4,5 $\rightarrow$E
7.	$\vert$ $\vert$ ${\sim}Ga$	H for $\exists$E
8.	$\vert$ $\vert$ $P \& {\sim}P$	6,7 CONT
9.	$\vert$ $P \& {\sim}P$	2, 7-8 $\exists$E
10.	${\sim}\forall xFx$	3-9 ~I

- This technique is often needed to finish an $\exists$E proof involving a contradiction.