Predicate Calculus

Rules for the Universal Quantifier

The Predicate Calculus

- Just as the language of predicate logic is an extension of the language of propositional logic, the predicate calculus is an extension of the propositional calculus.

- In addition to all the rules and equivalences of propositional logic, the predicate calculus contains four new rules for introducing and removing the quantifiers.

- Two of these rules concern the universal quantifier $\forall$.
  1. Universal Elimination.
  2. Universal Introduction.

- Two concern the existential quantifier $\exists$.
  1. Existential Introduction.
  2. Existential Elimination.

- These are all inference rules, not equivalences, so you can only apply them when the quantifier is the main operator in a formula.

Universal Elimination $\forall E$

- Universal elimination is a rule that allows you to remove the universal quantifier and substitute a name for the variable that it is binding.

- Universal elimination is what we call an unrestricted rule, Meaning that you may substitute any constant you like for the variable in question.

- The intuitive justification for this is straightforward. A universally quantified expression says that a certain property holds of every single thing in the domain of discourse.

- Hence, a statement like "Everyone is happy," $\forall xHx$, implies Ha, Hb, Hc, Hd…etc.

Example of $\forall$E

- Here is a simple example of how universal elimination works in a proof.

- $\forall xFx, \forall x(Fx \rightarrow Gx) \vdash Ga$

1. $\forall xFx$	A
2. $\forall x(Fx \rightarrow Gx)$	A
3. Fa	1, $\forall$E
4. $Fa \rightarrow Ga$	2, $\forall$E
5. Ga	3,4 $\rightarrow$E

- Notice that by creating lines 3 and 4, the rule of $\forall$E allows us to apply the familiar rule of $\rightarrow$E to get our conclusion.

- Also notice that to use this rule correctly we had to be sure to substitute the constant 'a' for every single variable bound by the universal quantifier. If we didn't do that, then we would have created an unbound variable, and the resulting expression would not be a wff..

Example of $\forall$E 2

- Here is an example involving multiple applications of $\forall$E.

- $\forall x \forall y(Fxy \rightarrow Fyx)$, $\forall xFax$ $\vdash$ Fda

1. $\forall x \forall y(Fxy \rightarrow Fyx)$	A
2. $\forall xFax$	A
3. $\forall y(Fay \rightarrow Fya)$	1, $\forall$E
4. Fad	2, $\forall$E
5. $Fad \rightarrow Fda$	3, $\forall$E
6. Fda	4,5 $\rightarrow$E

- So, notice here that we can only ever remove the main quantifier, which is the one furthest to the left. In the first assumption, we could not have removed $\forall y$ prior to removing $\forall x$.

- Also notice that on lines 4 and 5 we were able to take advantage of the fact that $\forall$E is an unrestricted rule by specifically choosing the name 'd' to substitute for the corresponding variable.

Example of ∀E 3

- Here is an example of an incorrect use of ∀E.
- ∀xFx v ∀yGy ├ Fa v Gb

1.	∀xFx v ∀yGy	A
2.	Fa v ∀yGy	1, ∀E
3.	Fa v Gb	2, ∀E

- Both lines 2 and 3 are incorrect because in each case the main operator is the disjunction, not the universal quantifier.
- This proof can be done, but you must use the rule of vE, as follows.

Example of ∀E 4

- ∀xFx v ∀yGy ├ Fa v Gb

1.	∀xFx v ∀yGy	A
2.	│ ∀xFx	H
3.	│ Fa	2, ∀E
4.	│ Fa v Gb	3, vI
5.	∀xFx → (Fa v Gb)	2-4 →I
6.	│ ∀yGy	H
7.	│ Gb	6, ∀E
8.	│ Fa v Gb	7, vI
9.	∀yGy → (Fa v Gb)	6-8, →I
10.	Fa v Gb	1,5, 9 vE

Example of ∀E 5

- The fact that you are dealing with universally quantified expressions does <u>not</u> automatically entail that you need to apply the rule of universal elimination. Sometimes the entire formula will satisfy the condition of a rule from propositional logic. For example:
- ~∀zGz → ~∀xFxa ├ ∀y∀xFxy → ∀zGz

1.	~∀zGz → ~∀xFxa	A
2.		∀y∀xFxy	H
3.		∀xFxa	2, ∀E
4.		~~∀xFxa	3, DN
5.		~~∀zGz	1,4 MT
6.		∀zGz	5, DN
7.	∀y∀xFxy → ∀zGz	2-6, →I

- The entire formula on line 4 is the negation of the consequent of the formula on line 1.

Formal statement of the rule of $\forall$E

- Here is the formal statement of the rule of $\forall$E.

- Rule of universal elimination
  - For any universally quantified formula $\forall x\mathcal{P}$, we may infer any formula that results from removing $\forall x$ and inserting a single name letter a for every occurrence of x in $\mathcal{P}$.

- To understand this definition properly you need to remember that the cursive letters are from the meta-language. $\mathcal{P}$ just refers to any partial formula at all; x is just some variable and a is just some name letter.

- The definition only makes sense if you understand $\mathcal{P}$ to contain the variable x. This is implied by the fact that it is bound by the quantifier $\forall x$; it is not represented explicitly, because x could occur in any number of ways and any number of times.

Universal Introduction $\forall$I

- The rule of universal introduction permits you to introduce, rather than eliminate, a quantifier. It is a restricted rule, meaning that you need to make sure that certain conditions have been satisfied before applying it. We will talk about those momentarily, but here is a simple $\forall$I proof to begin.

- $\forall xFx, \forall x(Fx \rightarrow Gx) \vdash \forall x\, Gx$

1. $\forall xFx$	A
2. $\forall x(Fx \rightarrow Gx)$	A
3. Fa	1, $\forall$E
4. $Fa \rightarrow Ga$	2, $\forall$E
5. Ga	3,4 $\rightarrow$E
6. $\forall x\, Gx$	5, $\forall$I

Universal Introduction restrictions

- The reason that ∀I is a restricted rule is because in most conditions it would be illegitimate to simply introduce a universal quantifier.
- For example, it would not be good if we could prove this.
- Fa ├ ∀xFx
- Obviously, it does not follow from the fact that Albert is funny, that everything is funny.
- But the reason we can sometimes justify introducing a universal quantifier, is that sometimes the name itself is just a place holder and doesn't really refer to any particular thing. We can see that this is the case in the proof we just did.

Universal Introduction restrictions 2

- $\forall x Fx, \forall x(Fx \rightarrow Gx) \vdash \forall x\, Gx$

1. $\forall x Fx$	A
2. $\forall x(Fx \rightarrow Gx)$	A
3. Fa	1, $\forall$E
4. $Fa \rightarrow Ga$	2, $\forall$E
5. Ga	3,4 $\rightarrow$E
6. $\forall x\, Gx$	5, $\forall$I

- Notice that in this case, 'a' on lines 3 and 4, could have been 'b' or 'c' or any name at all. We just needed some name so that we can do the operation on line 5.
- So we can do a universal introduction whenever the name in question is arbitrary or, put differently, could have been selected at random.

Universal Introduction ∀I restrictions 3

- The kind of arbitrariness we need to achieve in order for ∀I to deliver only valid results is achieved by the following two restrictions.

1. When we introduce a universal quantifier, the name for which the variable is being substituted must <u>not</u> occur in
   1. any <u>assumption</u>; or
   2. any <u>hypothesis</u> that has not yet been discharged.

2. The variable being introduced for a constant must be introduced for <u>every</u> occurrence of the constant in the formula.

Universal Introduction error 1

- Let's look at some examples of $\forall$I being done incorrectly.

- $\forall x Fx \rightarrow \forall x Gx,\ Fa \vdash \forall x Gx$

1. $\forall x(Fx \rightarrow Gx)$	A
2. Fa	A
3. $Fa \rightarrow Ga$	1, $\forall$E
4. Ga	2,3 $\rightarrow$E
5. $\forall x\, Gx$	4, $\forall$I

- Line 5 is not permissible because we are attempting to introduce the universal quantifier on the formula Ga, but the constant 'a' occurs in assumption 2.

- If you think about this, you can see that it is a good thing that we can not prove this. On one interpretation the premises say that if everything is funny, then everything is goofy. The fact that Albert is funny does not imply that everything is funny. So the conclusion that everything is goofy is unwarranted.

Universal Introduction error 2

- This botched proof violates the second restriction on ∀I.

- ∀ x∀y(Lxy v Sx) ├ Lre v Sb

1.	∀x∀y(Lxy v Sx)	A
2.	∀y(Lby v Sb)	1, ∀E
3.	Lbe v Sb	2, ∀E
4.	∀x(Lxe v Sb)	3, ∀I
5.	Lre v Sb	4, ∀E

- Line 4 is not permissible, because x needs to be substituted for every occurrence of b.

- Again, it is not too difficult to see that this is a bad inference. If L means 'loves' and 'S' means 'sad,' the assumption tells us that either every x loves every y, or that x is sad. From this we should be able to conclude that Rachel loves Elvis or Rachel is sad, but not that Rachel loves Elvis, or Bertha is sad.

One more proof involving $\forall$I

- Here is another correctly done proof involving $\forall$I.

- $\vdash \forall x \forall y(Fx \rightarrow (Gy \rightarrow (Fx \vee P)))$

1.	$\quad$ Fa	H
2.	$\quad\quad$ Gb	H
3.	$\quad\quad$ Fa v P	1, vI
4.	$\quad$ Gb $\rightarrow$ (Fa v P)	2-3, $\rightarrow$I
5.	Fa $\rightarrow$ (Gb $\rightarrow$ (Fa v P))	1-4, $\rightarrow$I
6.	$\forall$y(Fa $\rightarrow$(Gy $\rightarrow$ (Fa v P)))	5, $\forall$I
7.	$\forall$x$\forall$y(Fx $\rightarrow$(Gy $\rightarrow$ (Fx v P)))	6, $\forall$I

- This is a common way of proving universally quantified formulas. The proof might at first appear to break the rules, but notice that, while the names 'a' and 'b' do occur in the hypotheses, they have been discharged by the time we get to line 6, so we are abiding by the restrictions. If there had been assumptions in the proof, it would have been important to make sure that the constants we introduced in the hypotheses did not occur in the assumptions.
- (Also, notice that the P is not a predicate, but an atomic formula from propositional logic.)

Formal statement of the rule of ∀I

- Here is the formal statement of the rule of ∀I.

- <u>Rule of universal introduction</u>

  - For a formula $\mathcal{P}$ containing a name a we may infer any formula of the form $\forall x\mathcal{P}$, where the variable x is inserted for a in $\mathcal{P}$, as long as:
    1. a does not occur in <u>any</u> assumption <u>or</u> hypothesis in effect at the time of the substitution;
    2. the variable x is substituted for <u>every</u> occurrence of a in $\mathcal{P}$.

- Again, to understand this definition you need to remember that the cursive letters are from the meta-language. $\mathcal{P}$ just refers to any predicate logic formula containing a name letter,; x is just some variable and a is just some name letter.